Research Narrative

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1 Introduction

The following is a brief description of some of the multi-paper projects that I have been involved in recently. There are four sections (including this one), and the topics are as follows.

§2. The rank of hyperbolic 3-manifolds, see especially [22].

§3. The global topology of the Chabauty space of $\mathrm{PSL}_2(\mathbb{R})$, see [15] with Lazarovich and Leitner.

§4. My work at the intersection of geometry and measurable group theory, joint with various subsets of Abért, Bergeron, Gelander, Nikolov, Raimbault, Samet (abbreviated ABBGNRS when I am included as the second B), especially the paper with ABBG [1],

§5. Homeomorphisms of the boundary of a handlebody, e.g. [14, 17].

2 Rank, genus and carrier graphs

The *rank* of a closed 3-manifold M is the minimal number of elements needed to generate its fundamental group. Rank initially became popular through its connection with the *Heegaard genus* $g(M)$, the minimal genus of a surface $S \subset M$ that divides M into two handlebodies. Rank is at most genus, and in the 1960s, Waldhausen conjectured that the rank and Heegaard genus of a closed orientable 3-manifold are always equal. This was disproven by Boileau-Zieschang [25], see also Schultens-Weidmann [40] and Li [36], but in all known examples the Heegaard genus is at most twice the rank. Considerable effort has been made to either prove or disprove the existence of a linear bound, see e.g. [6, 35], but currently even the following conjecture is open:

Conjecture 2.1. *The Heegaard genus of a closed, orientable hyperbolic* 3*-manifold* M *is bounded above by some function of its rank.*

In another direction, inspired by an understanding of geometric limits of surface groups, McMullen asked whether the rank of M constrains its geometry in the following sense.

Question 2.2 (McMullen [10])**.** Is it true that the radius of the largest embedded ball in a closed hyperbolic 3-manifold M is bounded above by a function of $\mathrm{rank}(M)$?

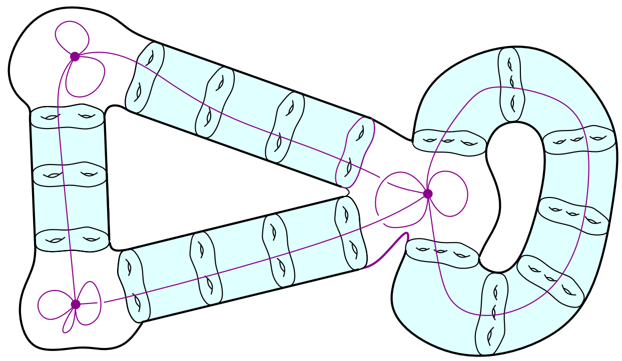

Figure 1: A geometric decomposition of a thick manifold with bounded rank into white building blocks and blue product regions, where the product regions may be very wide.

We note that Bachman-Cooper-White [8] have showed that a bound on Heegaard genus gives a bound on the largest embedded ball, so Conjecture 2.1 implies a positive answer to Question 2.2.

In a series of papers leading up to the final paper [22], joint with Souto, we show that both the Heegaard genus and the radius of the largest embedded ball in M can be bounded above by a function of $\text{rank}(M)$ and a lower bound for the injectivity radius $\text{inj}(M)$. This is a consequence of the following general geometric decomposition theorem.

Theorem 2.3 (B-Souto [22])**.** *Given k, ϵ, there are $n = n(k)$ and $D = D(k, \epsilon)$ as follows. Suppose that M is a closed hyperbolic 3-manifold M with injectivity radius at least ϵ and rank at most k. Then M has a geometric decomposition modeled on a graph with complexity at most n, where the vertices are 'building blocks' that are submanifolds of M with diameter bounded by D, and where the edges are 'wide product regions' foliated by level surfaces with diameter at most D.*

See Figure 1 for an illustration. To prove the theorem above, we represent generating sets for $\pi_1 M$ geometrically using minimal length *carrier graphs* in M, introduced by White [43] and further developed by myself and Souto [11, 21, 20, 41], and (very roughly) show that the geometry of M tracks the geometry of the carrier graph. The paper [22], which is 225 pages, was accepted by Memoirs of the AMS in April 2024.

2.1 Future directions and grad student projects

The obvious future direction in this program is to remove the dependence on injectivity radius from (a suitably modifed version of) Theorem 2.3. Souto and I already have one paper on the topic in which there is no assumption on injectivity radius needed. In [21] we prove:

Theorem 2.4 (B-Souto [21])**.** *If M is a closed hyperbolic 3-manifold that fibers over the circle with fiber Σ_g, and the monodromy map $\phi : \Sigma_g \longrightarrow \Sigma_g$ has translation distance in the curve complex $\mathcal{C}(\Sigma_g)$ at least some $C(g)$, then* $\text{rank}(M) = 2g + 1$.

As a first step, I plan to address the following conjecture.

Conjecture 2.5. *If (M_n) is a sequence of closed hyperbolic 3-manifolds with $\text{rank}(M_n) \leq k$ and $\text{vol}(M_n) \longrightarrow \infty$, then the Cheeger constants $h(M_n) \longrightarrow 0$.*

Under the assumption that $\text{inj}(M_n) \geq \epsilon > 0$, this is a theorem of mine with Souto [20] that was a precursor to Theorem 2.3. I expect similar arguments to apply, but the geometric limit arguments required are more complicated.

I also have a graduate student, Mujie Wang, who is working on a Heegaard splitting analogue of Theorem 2.4. Their goal is to prove that if M has a genus g Heegaard splitting with Hempel distance larger than some $C = C(g)$, then $\text{rank}(M) = g$. They have a proof sketch that shows this is the case asymptotically almost surely for 'random' M, i.e. M produced by gluing two copies of a genus g handlebody H together via a gluing map chosen according to a random walk on the mapping class group of ∂H.

3 The Chabauty space of $\mathrm{PSL}(2,\mathbb{R})$

Let G be a Lie group and let $\text{Sub}(G)$ be the set of all closed subgroups of G. We equip $\text{Sub}(G)$ with the *Chabauty topology*, in which $H_i \to H$ if all accumulation points of sequences $h_i \in H_i$ lie in H, and every $h \in H$ is a limit of a sequence $h_i \in H_i$. The global structure of $\text{Sub}(G)$ can be quite rich, even when G is relatively uncomplicated. For instance, Hubbard-Pourrezza [32] showed that when $G = (\mathbb{R}^2, +)$ the space $\text{Sub}(G)$ is homeomorphic to the 4-sphere S^4. Moreover, if we let $\mathcal{N} \subset \text{Sub}(G)$ be the set of all *non-lattices*, i.e. those subgroups of $\mathbb{R}^2$ that are not isomorphic to $\mathbb{Z}^2$, then $(\text{Sub}(G), \mathcal{N})$ is homeomorphic to the suspension of $(S^3, \text{trefoil knot})$.

When $G = \text{Isom}^+(\mathbb{H}^n)$, the Chabauty topology is of fundamental importance, especially in low dimensions, e.g. in Thurston's hyperbolization theorem for 3-manifolds. In this setting, Chabauty convergence of discrete groups is often rephrased in terms of the quotient orbifolds, as follows. If $\Gamma \in \text{Sub}(G)$ is discrete, the quotient $X := \Gamma \backslash \mathbb{H}^n$ is a hyperbolic n-orbifold, and if we fix a base frame[1] f_0 for $\mathbb{H}^n$, the projection $\bar{f}_0$ is a base frame for X, and

$$\{\Gamma \in \text{Sub}(G) \mid \Gamma \text{ discrete } \} \longrightarrow \left\{ \begin{smallmatrix} \text{oriented, framed hyperbolic} \\ n\text{-orbifolds } (X,f) \end{smallmatrix} \right\} \Big/ \text{o.p. framed isometry} \tag{3.1}$$

is a bijection that carries the Chabauty topology on the left to the *framed Gromov-Hausdorff topology* on the right, in which $(X, f), (X', f')$ are close if there is an o.p. almost-isometry between large neighborhoods of the base frames that takes one base frame to the other.

However, although much is known about convergence of framed hyperbolic manifolds, the global structure of $\text{Sub}(G)$ has not been extensively studied. With Nir Lazarovich, Arielle Leitner, and also Sangsan Warakkagun, I have been trying to understand the structure of $\text{Sub}(G)$ when

$$G = \mathrm{PSL}(2,\mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2).$$

First, some important subspaces of $\text{Sub}(G)$ are as follows.

1. The topology of the subspace $\text{Sub}_{elem}(G)$ of elementary subgroups can be completely described, see BLL [15]. In particular, it is homotopy equivalent to the union of a certain countable set of spheres in $\mathbb{R}^3$.

2. For each finite type, oriented topological 2-orbifold S, there is a subset $\text{Sub}_S(G) \subset \text{Sub}(G)$ consisting of all groups $\Gamma < G$ such that $\Gamma \backslash \mathbb{H}^2$ is a finite volume 2-orbifold that is o.p.

[1] A base frame for M is an orthonormal frame for the tangent space TM_p at some point $p \in M$.

isometric to S. Via the correspondence in (3.1), $\mathrm{Sub}_S(G)$ can be thought of as the moduli space of all framed hyperbolic structures on S, and it (orbi)-fibers over the traditional moduli space $\mathcal{M}(S)$ with (regular) fibers homeomorphic to the frame bundle of S. See BLL [15].

In a 2-paper collaboration with Lazarovich and Leitner, the first being [15] and the second in preparation, we understand connectivity properties of $\mathrm{Sub}(G)$ as follows. First, the connected components of $\mathrm{Sub}(G)$ are the singleton set $\{G\}$, the spaces $\mathrm{Sub}_S(G)$ where S is a sphere with three cone points, and the rest of $\mathrm{Sub}(G)$, which we'll call $\mathrm{Sub}^1(G)$ here, since it's the connected component of the identity. So, $\mathrm{Sub}^1(G)$ contains basically all the interesting topology of $\mathrm{Sub}(G)$.

The main goal of the two papers, though, is to compute the fundamental group of $\mathrm{Sub}^1(G)$.

Conjecture 3.1 (BLL)**.** *$\pi_1 \mathrm{Sub}^1(G)$ is an infinitely generated free group.*

The nontrivial elements of π_1 come from attaching certain multiply-ended spaces $\mathrm{Sub}_S(G)$ to the rest of $\mathrm{Sub}^1(G)$; these S are all spheres with 4 cone points. In the first paper [15], we give a fine analysis of how the spaces $\mathrm{Sub}_S(G)$ are attached to each other, proving for instance that their boundaries have neighborhood deformation retracts. In the second paper, we'll focus on the space

$$\mathrm{Sub}_\infty(G) := \{ \text{ discrete } \Gamma < G \text{ with infinite covolume}\}.$$

One of the main results will be the following.

Theorem 3.2 (BLL, in preparation)**.** *The space $\mathrm{Sub}_\infty(G)$ deformation retracts onto the subset $\mathrm{Sub}_{fin}(G)$ consisting of all finite subgroups of G.*

An attractive special case of the argument, phrased in terms of quotients, says that the space of all infinite volume framed hyperbolic surfaces is contractible. Intuitively, one contracts an arbitrary framed surface to $\mathbb{H}^2$ by deforming the metric so that the injectivity radius at the base frame goes to infinity. However, it's subtle to do this in a canonical enough way to get a continuous contraction! We end up using a partition of unity to patch together a vector field v on $\mathrm{Sub}_\infty(G)$ (which is not a manifold, but is almost a foliated space) such that flowing along v gives the contraction.

In a separate project with Sangsan Warakkagun, my former student, I have been studying local connectivity of $\mathrm{Sub}(G)$. This is a natural question, since there's a way in which the subset of $\mathrm{Sub}(G)$ consisting of infinitely generated groups has a sort of fractal structure. While $\mathrm{Sub}(G)$ is not locally connected, it turns out that the only problem is groups with unbounded torsion.

Theorem 3.3 (BW, in preparation)**.** *If $m > 0$, let $\mathrm{Sub}^m(G)$ be the set of closed subgroups $\Gamma < G$ such that every finite order element of Γ has order at most m. Then $\mathrm{Sub}^m(G)$ is locally connected.*

Finally, my graduate student Matt Zevenbergen has been studying the topology of $\mathrm{Sub}(G)$ when $G = \mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$. Here, the picture is quite different. Each finite volume hyperbolic 3-orbifold M determines a component of $\mathrm{Sub}(G)$, so all the interesting topology comes from infinite volume orbifolds. Disregarding torsion for simplicity, let's define

$$\mathcal{H}^3_\infty := \{ \text{ framed infinite volume hyperbolic 3-manifolds } \}/\text{framed isometry}.$$

Using the big theorems from Kleinian groups, Matt can show that $\mathcal{H}^3_\infty$ is connected. But surprisingly, it is not path connected! Matt shows that if $\mathbb{Z}$-many copies of a compact hyperbolic 3-manifold with two isometric totally geodesic boundary components are glued end-to-end to create a hyperbolic 3-manifold M, then $\{(M, f) \mid f \text{ a frame in } M\}$ is a path component of $\mathcal{H}^3_\infty$.

4 Benjamini-Schramm convergence

In this section we'll discuss some applications of 'Benjamini-Schramm (BS) convergence' of sequences of finite volume Riemannian manifolds, mostly from my papers [1, 2, 4, 5, 18]. Let

$$\mathcal{M}^d = \{\text{pointed Riemannian } d\text{-manifolds } (M,p)\}/\text{pointed isometry}.$$

We consider $\mathcal{M}^d$ with the *smooth topology,* where (M,p) and (N,q) are close if their base points are contained deep inside compact subsets of M and N that are diffeomorphic, and on which the Riemannian metrics are smoothly close, see [5, 39]. For each fixed manifold M, there is a map

$$M \longrightarrow \mathcal{M}^d, \quad p \longmapsto (M,p), \tag{4.1}$$

and when M has finite volume, we let μ_M be the finite measure on $\mathcal{M}^d$ obtained by pushing forward the Riemannian measure vol on M under (4.1).

Definition 4.1 (BS-convergence)**.** We say that a sequence of finite volume Riemannian d-manifolds (M_n) *BS-converges* if the probability measures $(\mu_{M_n}/\operatorname{vol}(M_n))$ weak* converge, in which case the *BS-limit* is the limiting probability measure μ on $\mathcal{M}^d$.

Intuitively, the BS-limit measure μ encodes (for large n) what M_n looks like near a randomly chosen base point: namely, if you select a μ-random (M,p) and take a picture of some bounded neighborhood of the base point p, the distribution of pictures you'll get is close to what you see around randomly chosen base points for M_n.

Benjamini-Schramm (BS) convergence was first considered in 2001 by Benjamini-Schramm [9] in the setting of finite graphs. In the paper ABBGNRS [3], my coauthors and I adapted it to sequences of locally symmetric spaces, although in that paper we focused more on an equivalent algebraic interpretation of BS-convergence, in which sequences (Γ_n) of lattices in a fixed Lie group G converge to *invariant random subgroups* of G, since studied by myself [12, 23] and many others. The precise definition above comes from my paper [5], in which Abért and I develop some foundational properties of BS-limits in the Riemannian setting.

The measures μ_M above and their weak* limits μ are examples of *unimodular* measures on $\mathcal{M}^d$, see [5] for the definition. Informally, unimodularity means that for a given manifold M, the μ-probability of selecting the pointed manifold (M,p) is the same for all p. For another perspective, if we vary M, the images of the maps in (4.1) form the 'leaves' of a 'foliation' of $\mathcal{M}^d$, and unimodular μ are the *completely invariant* measures of this foliation, i.e. measures that are obtained by integrating the Riemannian volumes on the leaves against some transverse measure. Now, the leaves here are singular — they are of the form $M/\operatorname{Isom}(M)$ and may not be manifolds — so this is not an actual foliation. One of the main theorems in [5] is that one can make this viewpoint precise by lifting to a foliated *desingularization* of $\mathcal{M}^d$ in which the problematic symmetries that collapse the leaves have been blocked, and where unimodular measures on $\mathcal{M}^d$ lift to completely invariant measures.

Here are some applications of BS-convergence from my papers [1, 2]. Let X be an irreducible Riemannian symmetric space of noncompact type. Then X has transitive isometry group, so because of the equivalence relation on $\mathcal{M}^d$, the image of X under the map in (4.1) is a point. We say that a sequences of locally symmetric quotients $M_n = \Gamma_n \backslash X$ *BS-converges to* X if the BS-limit is an atomic measure on this point. This happens when for large n, M_n looks much like X near a randomly chosen basepoint, which is equivalent to the condition that for all $R > 0$,

$$\frac{\operatorname{vol}\{p \in M_n \mid \operatorname{inj}_{M_n}(p) < R\}}{\operatorname{vol}(M_n)} \to 0.$$

In [2], my coauthors and I noticed that combining this framework with the Stuck-Zimmer Theorem [42] from ergodic theory has the following strong consequence.

Theorem 4.2 (ABBGNRS [2])**.** *If* $\operatorname{rank}_{\mathbb{R}}(X) \geq 2$ *then any sequence of pairwise non-isometric, finite volume quotients* $M_n = \Gamma_n \backslash X$ *BS-converges to* X*.*

We then applied this observation to study the growth of the Betti numbers $b_k(M_n)$.

Theorem 4.3 (ABBGNRS [2])**.** *If* (M_n) *BS-converges to* X *and for some fixed* $\epsilon > 0$ *we have* $\operatorname{inj}(M_n) > \epsilon$*, then for all* k *the volume-normalized Betti numbers*

$$\frac{b_k(M_n)}{\operatorname{vol}(M_n)} \to \beta_k^{(2)}(X).$$

Here, $\beta_k^{(2)}(X)$ is the k^{th} L^2-Betti number of X. The above was known previously for (M_n) that are covering towers, by work of DeGeorge and Wallach [28], see also Lück [37], but this was the first really general result that was uniform over all quotients of a fixed symmetric space.

In the later paper ABBG [1], my coauthors and I removed the assumption on injectivity radius from Theorem 4.3. In fact, we proved the following more general result:

Theorem 4.4 (ABBG [1])**.** *Suppose* $X \neq \mathbb{H}^3$ *and* $M_n = \Gamma_n \backslash X$ *is a BS-convergent sequence of finite volume quotients. Then the sequence* $b_k(M_n)/\operatorname{vol}(M_n)$ *converges.*

It follows that if $M_n \to X$, then $b_k(M_n)/\operatorname{vol}(M_n) \to \beta_k^{(2)}(X)$. One can see this by interleaving the sequence (M_n) with another sequence that BS-converges to X where the limit of normalized Betti numbers is already known, e.g. a covering tower, and then applying Theorem 4.4. The assumption $X \neq \mathbb{H}^3$ above is necessary, essentially because Dehn filling can dramatically alter Betti numbers without appreciably changing geometry.

We gave two proofs of Theorem 4.3 in [2], one representation theoretic and one via an analysis of heat kernels. The proof of Theorem 4.4 is very different. Essentially, we rely on a similar result of Elék [29] that says that if (K_n) is a BS-convergent sequence of simplicial complexes with bounded degree, then $b_k(K_n)/\operatorname{vol}(K_n)$ converges for all k. Elek had suggested that one could apply this to sequences of Riemannian manifolds with an injectivity radius lower bound by using a random process to turn the manifolds into simplicial complexes (take the nerve complex associated to a net constructed via Poisson processes). Bowen [26] implemented this argument, but his proof had an error. We fixed the error, applied this argument to the thick parts of the (M_n), and then showed that adding back in the thin parts doesn't change the volume-normalized Betti numbers much if $X \neq \mathbb{H}^3$. Most of the difficulty in the paper comes from analyzing how the Elek/Bowen argument interfaces with the shape of the boundary of the thick part, which can be complicated.

I have a few other papers on this topic. In a second ABBGNRS collaboration [4], we construct many interesting examples of BS-limits of sequences of hyperbolic manifolds that show how flexible the situation in rank 1 is, in contrast to Theorem 4.2. In [19], Raimbault and I show that the end spaces of 'unimodular random manifolds', i.e. random manifolds that appear as BS-limits, are very regular, for instance there are either $0, 1, 2$ or a Cantor set of them. The papers [24, 13], involve the algebraic version of BS-limits, namely invariant random subgroups of Lie groups. In the first paper, Tamuz and I study their 'unimodularity' properties, and in the second, Bowen, Tamuz and I classify invariant random subgroups of certain semidirect products.

More recently, I have been interested in applying these techniques to study the growth of rank with respect to volume. A celebrated recent result of Fraczyk-Mellick-Wilkens [30] states that if $M_n = \Gamma_n \backslash X$ is a sequence of finite volume quotients of a higher rank symmetric space X, and (M_n) BS-converges to X, then the *rank gradient*

$$RG(M_n) := \lim_{n\to\infty} \operatorname{rank}(M_n)/\operatorname{vol}(M_n)$$

is zero. Here, as in §2, rank refers to the minimal number of generators for π_1. The question of whether the corresponding statement is true for $X = \mathbb{H}^3$ is of considerable interest. For instance:

1. There is a sequence (M_n) of arithmetic hyperbolic 3-manifolds where $M_n \to \mathbb{H}^3$ but

   $$\lim_{n\to\infty} g(M_n)/\operatorname{vol}(M_n) > 0,$$

   where g is Heegaard genus, see e.g. [7, Pf of Theorem 2, pg 21]. So, if $\operatorname{rank}(M_n)/\operatorname{vol}(M_n) \to 0$, there is a sequence of hyperbolic 3-manifolds where rank grows sublinearly in volume while Heegaard genus grows linearly, which is a very strong counterexample to Waldhausen's conjecture that rank and genus are equal. (In the counterexamples [25, 40, 36] cited in §2, rank and genus are always within a factor of 2.)

2. On the other hand, there are examples of sequences of hyperbolic 3-manifolds (M_n) with $M_n \to \mathbb{H}^3$ and where $RG(M_n)$. If one can also construct such examples where $RG \neq 0$, one can show that Gaboriau's *Fixed Price Conjecture* in measurable dynamics is wrong.

Inspired by this discussion, I propose the following conjecture.

Conjecture 4.5. *Suppose (M_n) is a sequence of hyperbolic 3-manifolds with $RG(M_n) = 0$. Then*

$$h(M_n)/\operatorname{vol}(M_n) \to 0,$$

where $h(\cdot)$ is the Cheeger constant.

The sequence of examples mentioned in 1. above have $h(M_n)/\operatorname{vol}(M_n)$ bounded below, so if the conjecture is true, then this sequence has nonzero rank gradient, answering the question above.

Note the similarity between Conjectures 2.5 and 4.5. They are both direct generalizations of my work with Souto in [20], but in Conjecture 2.5 the point is to allow $\operatorname{inj}(M_n) \to 0$, while in Conjecture 4.5 the point is to replace a uniform upper bound on rank with a sublinear bound on rank. We note that it is fine in all our desired applications to only prove Conjecture 4.5 for sequences M_n with $\operatorname{inj}(M_n) \geq \epsilon > 0$, so the two conjectures are independent in spirit. Briefly, the idea for Conjecture 4.5 is to combine some of the carrier graph machinery from §2 with certain distributional limits in the style of Benjamini-Schramm.

5 The geometry of handlebodies

Let H be a genus g handlebody and let $S = \partial H$. Part of my research involves studying how the topology of S (its self-homeomorphisms, its geodesic laminations, etc...) interacts with the structure of H. For instance, we say that $f : S \longrightarrow S$ *extends* to H if it is the boundary restriction of a homeomorphism of H, and the *handlebody group* is defined to be

$$\mathrm{MCG}(H) = \{ \text{ homeomorphisms } f : S \longrightarrow S \text{ that extend to } H \ \}/\text{isotopy} \subset \mathrm{MCG}(S)$$

In [38], Masur introduced what is now called the *limit set* of H, defined as

$$\Lambda(H) = \overline{\{\text{meridians } \gamma \subset S\}} \subset \mathcal{PML}(S),$$

where a *meridian* is an essential simple closed curve on S that bounds a disk in H. This $\Lambda(H)$ acts as a dynamical limit set for the action $\mathrm{MCG}(H) \curvearrowright \mathcal{PML}(S)$. In some ways, $\Lambda(H)$ is similar to the limit set of a geometrically finite Kleinian group: for instance, Kerckhoff [33] and Gadre [31] show that $\Lambda(H)$ has measure zero. However, it is still quite mysterious! For instance:

Question 5.1. Suppose $\lambda, \mu \in \mathcal{PML}(S)$ have the same support, i.e. they are two transverse measures on the same geodesic lamination. If $\lambda \in \Lambda(H)$, is $\mu \in \Lambda(H)$?

The answer is unknown. However, Sebastian Hensel and I can reduce the question to the case that λ, μ are minimal and filling, and we can show that the answer is 'yes' when μ is ergodic.

If f is a pseudo-Anosov element of $\mathrm{MCG}(H)$, then the attracting and repelling laminations $\lambda_\pm(f)$ lie in $\Lambda(H)$. While the converse is not true, with Johnson and Minsky I showed:

Theorem 5.2 (B–Johnson–Minsky [14])**.** *If f is pseudo-Anosov then $\lambda_+(f) \in \Lambda(H)$ if and only if $\lambda_-(f) \in \Lambda(H)$, and this occurs if and only if some power f^n extends to a homeomorphism of some nontrivial subcompression body of H.*

Here, a *subcompression body* of H is a 3-submanifold of H obtained by choosing a finite union $\mathcal{D}$ of disjoint properly embedded discs in H, taking a regular neighborhood of $S \cup \mathcal{D}$ within H, and adding in any complementary components that are balls. One reason why Theorem 5.2 is interesting is that it reconcile two competing notions of genericity used to ensure hyperbolicity of certain gluings of 3-manifolds, see Lackenby [34].

In the recent paper [17], Cyril Lecuire and I give a fine study of Hausdorff limits of meridians and an associated technical tool called 'homoclinic leaves'. As a corollary of this and some of the arguments in the paper [14] referenced above, we characterize extension of *reducible* $f \in \mathrm{MCG}(S)$ to subcompression bodies of H. After passing to a power, a reducible element leaves invariant a decomposition $S = S_1 \cup \cdots \cup S_k$, where for each i, the restriction $f_i|_{S_i}$ is either the identity, a Dehn twist (if S_i is an annulus) or a pseudo-Anosov on S_i. We show that extension of f can be reduced to certain extension properties of the f_i, individually and in pairs. See [17, Theorem 1.6].

The proof of Theorems 5.2 uses hyperbolic geometry: we analyze the algebraic and geometric limits of certain sequences of convex cocompact hyperbolic structures on *int*(H). In a forthcoming paper, Lecuire and I [16] extend this analysis to a description of the geometric limits of *iterations in Schottky space*, proving a compressible version of Jeff Brock's thesis [27].

Recall that in his thesis, Brock studied *iterations in a Bers slice*, in which one fixes a homeomorphism $f : S \longrightarrow S$ and two points $X, Y \in \mathcal{T}(S)$ in the Teichmüller space of a closed surface S, and study the geometric limiting behavior of the hyperbolic 3-manifolds $M_n = Q(f^n(X), Y)$, which are homeomorphic to $S \times \mathbb{R}$ and have two conformal boundary components identified with $f^n(X)$ and Y, respectively. Here, if f is reducible and preserves a decomposition $S = S_1 \cup \cdots \cup S_k$

then typically what happens is that (M_n) converges to a manifold obtained from $S \times \mathbb{R}$ by drilling out the union of all the pseudo-Anosov and Dehn twist components S_i from $S \times \{0\} \subset S \times \mathbb{R}$.

To do *iteration in Schottky space*, we let S be the boundary of a handlebody H, we fix $f : S \longrightarrow S$ and a point $X \in \mathcal{T}(S)$ in Teichmüller space, and we consider the sequence

$$M_n := Q(f^n(X))$$

of convex-cocompact hyperbolic structures on H, where the conformal boundary of M_n is $f^n(X)$.

Theorem 5.3 (B-Lecuire, in preparation). *One can describe all geometric limits of* (M_n).

It takes a few pages in our paper to properly state our theorem, so I'll omit the precise statement here. However, here are some of the subtleties that come up. First, if f extends to a subcompression body $C \subset H$, then the Nielsen-Thurston type of f is irrelevant to the behavior of (M_n); rather, what matters is the Nielsen-Thurston type of the induced maps on the interior boundary components of C. Also, there are some subtle issues that arise if two of the invariant subsurfaces of f, say S_i and S_j, bound an interval bundle in H, or if boundary components of S_i and S_j bound an annulus in H. At the end of the day, one gets that the geometric limits are obtained by deleting certain surfaces from H, in analogy to Brock's theorem, but it's more complicated to describe exactly what to delete, since what matters is not just the homeomorphism f, but how f interacts with the structure of the handlebody H.

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